

HÖLDER REGULARITY FOR NONDIVERGENCE NONLOCAL PARABOLIC EQUATIONS

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ABSTRACT. This paper proves Hölder continuity of viscosity solutions to certain nonlocal parabolic equations that involve a generalized fractional time derivative of Marchaud or Caputo type. As a necessary and preliminary result, this paper first shows that viscosity solutions to certain nonlinear ordinary differential equations involving the generalized fractional time derivative are Hölder continuous.

1. INTRODUCTION

This paper studies nonlocal parabolic equations of nondivergence type involving a generalized fractional time derivative. Specifically, we study

$$\begin{aligned} f(x, t) &= \mathcal{D}_t^\alpha u(x, t) - \mathcal{I}u(x, t) \\ &:= \int_{-\infty}^t [u(x, t) - u(x, s)] \mathcal{K}(t, s) \, ds \\ &\quad - \sup_i \inf_j \left(\int_{\mathbb{R}^n} [u(x + y, t) + u(x - y, t) - 2u(x, t)] K_{ij}(x, y, t) \right). \end{aligned} \quad (1.1)$$

The main result is that viscosity solutions are Hölder continuous. Before stating the exact assumptions made for the kernels \mathcal{K} and K_{ij} , we first describe the recent history of showing Hölder continuity for parabolic equations of nondivergence type as well as the motivation for studying a parabolic equation involving a generalized fractional time derivative.

For local linear parabolic equations of nondivergence type, Krylov and Safonov [11] proved Hölder continuity of solutions without requiring any regularity assumptions on the coefficients of the equation. This method was later adapted to study regularity properties for fully nonlinear elliptic equations [6]. L. Wang [16] adapted the methods for nonlinear elliptic equations to prove the continuity of solutions to fully nonlinear parabolic equations. In recent years nonlocal equations have gained interest due to the applications in the physical sciences [13, 17]. Of particular interest are nonlocal elliptic operators of integro-differential type

$$\mathcal{L}u(x) = \int_{\mathbb{R}^n} [u(x) - u(y)] K(x, y). \quad (1.2)$$

When $K(x, y) = c_{n, \sigma} |x - y|^{-n-2\sigma}$, then \mathcal{L} is simply the fractional Laplacian $(-\Delta)^\sigma$. An assumption such as $K(x, y) = K(x, -y)$ will make \mathcal{L} a nondivergence type operator whereas an assumption such as $K(x, y) = K(y, x)$ will give an operator of

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divergence type. If \mathcal{L} is of nondivergence type, then \mathcal{L} is the nonlocal analogue of the linear elliptic operator

$$Lu = a_{ij}u_iu_j.$$

A typical ellipticity assumption on $K(x, y)$ is that $\lambda|x - y|^{-n-2\sigma} \leq K(x, y) \leq \Lambda|x - y|^{-n-2\sigma}$, and is analogous to the assumption for local equations that $\lambda|\xi|^2 \leq a_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2$. Although regularity estimates were known for operators of type (1.2), these estimates were not uniform as the order σ of the operator went to 1. Caffarelli and Silvestre [5] adapted the techniques for fully nonlinear local equations [6] to the nonlocal setting, and obtained uniform estimates as the order of the operator σ went to 1. Chang Lara and Davila [12] then proved regularity estimates for parabolic equations of the type

$$\partial_t u(x, t) - \mathcal{I}u(x, t) = 0,$$

where \mathcal{I} is a nonlocal nonlinear elliptic operator. The estimates in [12] are uniform as the order of the operator σ approaches 1, so that the results in [12] are not only a nonlocal analogue, but also recover many of the regularity results for local parabolic equations.

The equation of interest in this paper is motivated by the following equation

$$\partial_t^\alpha u - D_{|x|}^\beta u = f, \tag{1.3}$$

introduced in [7, 8] to model plasma transport. The function u is the probability density function for tracer particles in the plasma which represents the probability of finding a particle at time t and position x . The right hand side f is a source term. The nonlocal diffusion operator $D_{|x|}^\beta$ is one-dimensional and accounts for avalanche-like transport. The fractional derivative ∂_t^α is the Caputo derivative, and in the context of this model, the Caputo derivative accounts for the trapping of the trace particles in turbulent eddies. Although (1.3) is an equation for one spatial dimension, we may consider the following nonlocal parabolic equation in higher dimensions

$$\partial_t^\alpha u - \mathcal{L}u = f \tag{1.4}$$

where \mathcal{L} is nonlocal elliptic operator of type (1.2), and ∂_t^α is the Caputo derivative. As mentioned earlier, certain assumptions made on the kernel $K(x, y)$ lead to an operator of either divergence or nondivergence type. The author in [1] studied weak solutions of (1.4) of nondivergence type. The assumptions made on the kernel $K(x, y, t)$ were that

$$\frac{\lambda}{|x - y|^{n+\sigma}} \leq K(x, y, t) \leq \frac{\Lambda}{|x - y|^{n+\sigma}}, \tag{1.5}$$

for fixed $0 < \lambda \leq \Lambda$, and that $K(x, y, t) = K(x, -y, t)$. For divergence form equations, the author in [2] studied weak solutions of (1.4) of divergence type and proved a De Giorgi-Nash-Moser type theorem which gives Hölder continuity of solutions. The results in [2] assumed (1.5) and $K(x, y, t) = K(y, x, t)$. The latter assumption leads to (1.4) being an equation of divergence type. The author showed in [3] that the methods utilized in [2] can be applied to an equation of type (1.4) with a much more general fractional time derivative. The main aim of this paper is to extend the results for nondivergence form equations in [1] to equations involving this more general fractional time derivative which we now describe.

1.1. The Marchaud and Caputo Derivatives. The Caputo derivative is useful in modeling phenomena that take into account interactions in the past. In this manner one can think of the equation has having “memory”. This is in contrast to parabolic equations such as the heat operator $\partial_t - \Delta$ that do not account for the past. The Caputo derivative is defined as

$${}_a\partial_t^\alpha f(t) := \frac{1}{\Gamma(1-\alpha)} \int_a^t \frac{f'(s)}{(t-s)^{1+\alpha}} ds.$$

For C^1 functions one may use integration by parts to show the equivalent formula

$${}_a\partial_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \frac{f(t) - f(a)}{(t-a)^\alpha} + \frac{\alpha}{\Gamma(1-\alpha)} \int_a^t \frac{u(t) - u(s)}{(t-s)^{1+\alpha}} ds.$$

If we define $f(t) = f(a)$ for $t < a$ then as in [2] we have the equivalent formulation

$$\partial_t^\alpha f(t) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{f(t) - f(s)}{(t-s)^{1+\alpha}} ds. \quad (1.6)$$

This one-sided nonlocal derivative is known as the Marchaud derivative [14], and was recently studied in [4]. The formulation in (1.6) is very useful. It is no longer essential to know, and therefore label, the initial point a . Another useful feature of this formulation in (1.6) is rather than assigning initial data as simply $u(a) = c$, one may assign more general “initial” data as $u(t) = \phi(t)$ for $t \leq a$ with $\phi(x)$ not necessarily differentiable or even continuous. The formulation in (1.6) will be particularly useful for the notion of viscosity solutions in the context of nondivergence solutions later described in Section 2. The formulation in (1.6) looks similar to the one-dimensional fractional Laplacian except that the integration occurs from only one side. We may then apply many of the techniques developed for nonlocal elliptic operators like the fractional Laplacian to equations involving the Caputo derivative as was done in [2, 3]. Finally, this formulation in (1.6) allows for a different type of generalization of the Caputo derivative. Rather than generalizing as

$$\frac{d}{dt}(k * (f - f(a))) \quad \text{or} \quad \int_a^t f'(s)K(t-s) ds,$$

one may generalize as

$$\mathcal{D}_t^\alpha f(t) := \int_{-\infty}^t [f(t) - f(s)]\mathcal{K}(t, s) ds. \quad (1.7)$$

The proof of Hölder continuity in [2] for the linear divergence equation works for the more general fractional time derivative

$$\int_{-\infty}^t [f(t) - f(s)]\mathcal{K}(t, s, x),$$

provided that the kernel satisfies

$$\mathcal{K}(t, t-s, x) = \mathcal{K}(t+s, t, x) \quad (1.8)$$

and

$$\frac{\alpha}{\Gamma(1-\alpha)} \frac{\lambda}{(t-s)^{1+\alpha}} \leq \mathcal{K}(t, s, x) \leq \frac{\alpha}{\Gamma(1-\alpha)} \frac{\Lambda}{(t-s)^{1+\alpha}}. \quad (1.9)$$

See for instance [3] where a kernel $\mathcal{K}(t, s, x)$ satisfying only (1.8) and (1.9) is utilized. The assumption (1.9) is analogous to the assumption (1.5) for the kernel $K(x, y, t)$

of \mathcal{L} . Likewise, the condition (1.8) is analogous to $K(x, y, t) = K(y, x, t)$ and was necessary in [3] because the equation was of divergence form.

1.2. Main Results. We will assume that the kernels K_{ij} satisfy (1.5) and $K_{ij}(x, y, t) = K_{ij}(x, -y, t)$. We will also assume that the kernel $\mathcal{K}(t, s)$ satisfies (1.9).

In order to prove Hölder continuity of solutions to nondivergence parabolic equations involving the Caputo derivative, the author in [1] followed the idea in [15] to solve an ordinary differential equation in order to capture information backwards in time for a solution. In the context of solutions to (1.1) this requires solving

$$\begin{cases} m(t) = 0 & \text{if } t \leq -2 \\ \mathcal{D}_t^\alpha m(t) = c_0 |\{x \in B_1 : u(x, t) \leq 0\}| - C_1 m(t) & \text{if } t \in (-2, 0). \end{cases} \quad (1.10)$$

Several complications arise when considering solutions of (1.10). First, we need to prove existence of solutions. Second, the right hand side is not necessarily continuous. Third, a solution of (1.10) may not be regular enough to utilize as a part of a test function for the notion of viscosity solution. Fourth, we need to show roughly that if $|\{x \in B_1 \times (-2, -1) : u(x, t) \leq 0\}| \geq \mu_1$, then $m(t) \geq \mu_2$ if $t \in (-1, 0)$. When $\mathcal{D}_t^\alpha = \partial_t^\alpha$ the author in [1] utilized explicit representation formulas [9] for solutions to (1.10) to obtain the necessary properties. Since no such formulas are available for solutions to (1.10) in general, we overcome the four mentioned complications in a different way. We first show a priori Hölder continuity estimates for such ordinary differential equations. Our class of weak solutions will be considered in the viscosity sense as described in Section 2. Our first main result is

Theorem 1.1. *Let f be a continuous function on $[-1, 0]$, and let u be a bounded and a viscosity solution to*

$$\sup_k \inf_l ([u(x, t) - u(x, s)] \mathcal{K}^{kl}(t, s) ds) = f(t)$$

on $(-1, 0]$ with $f \in L^\infty$. Assume also that the kernels $\mathcal{K}^{kl}(t, s)$ satisfy (1.9). Then if $0 < \alpha \leq 1$ there exists two constants $0 < \beta \leq 1$ and $C > 0$ depending on α, λ, Λ and $\|u\|_{L^\infty((-\infty, -1))}$ but uniform as $\alpha \rightarrow 1$ such that

$$\|u\|_{C^{0, \beta}([-1/2, 0])} \leq C \|f\|_{L^\infty}. \quad (1.11)$$

We utilize Theorem 1.1 to prove existence of solutions to (1.10). To accommodate the second and third complications we show in Section 5 that we may approximate (1.10) uniformly from below by Lipschitz subsolutions to (1.10) which will be sufficient for the purposes of this paper. The fourth complication, however, remains. The Hölder estimate in (1.11) depending on an L^∞ norm of f is insufficient to overcome the fourth complication. We therefore consider a subclass of kernels $\mathcal{K}(t, s)$ which we assume also satisfy (1.8). This last assumption allows us to prove our second main result.

Theorem 1.2. *Let u be a bounded, continuous viscosity solution to (1.1) in $B_2 \times [-2, 0]$. Assume the kernels $K_{ij}(x, y, t)$ satisfy $K_{ij}(x, y, t) = K_{ij}(x, -y, t)$ and (1.5). Assume the kernel $\mathcal{K}(t, s)$ satisfies (1.9) and (1.8). Then there exists three constants $C, \kappa, \epsilon_0 > 0$ depending only on $\Lambda, \lambda, n, \alpha, \sigma$, but uniform as $\alpha \rightarrow 1$ such that u is Hölder continuous in $B_1 \times [-1, 0]$, and for $(x, t), (y, s) \in B_1 \times [-1, 0]$ the following estimates holds*

$$|u(x, t) - u(y, s)| \leq C(\|u\|_{L^\infty} + \epsilon_0^{-1} \|f\|_{L^\infty}) |x - y|^\kappa + |t - s|^{\kappa\alpha/(2\sigma)}.$$

We note that the estimates in Theorem 1.2 remain uniform as the order α of the fractional time derivative approaches 1. However, the estimates do not remain uniform as the order of the elliptic operator σ approaches 1. Having obtained the necessary theory for solutions to ordinary differential equations involving \mathcal{D}_t^α , the author plans in a future work to utilize the techniques of Chang Lara and Davila in [12] to prove Theorem 1.2 with estimates uniform as σ approaches 1. What would also be of interest is to prove Hölder regularity to solutions of an equation of the form

$$\begin{aligned} & \sup_k \inf_l \left(\int_{-\infty}^t [u(x, t) - u(x, s)] \mathcal{K}^{kl}(t, s, x) \right) \\ & - \sup_i \inf_j \left(\int_{\mathbb{R}^n} u(x + y, t) - u(x, t) K^{ij}(t, x, y) \right) = f(t, x). \end{aligned} \quad (1.12)$$

To do so it appears necessary to prove Theorem 1.1 with the Hölder estimate depending on the L^p norm of the right hand side f .

1.3. Notation. We here define notation that will be consistent throughout the paper.

- ∂_t^α - the Caputo derivative as defined in (1.6).
- \mathcal{D}_t^α - the generalized Marchaud derivative as defined in (1.7)
- α - will always denote the order of the fractional time derivative.
- σ - will always denote the order of the nonlocal elliptic spatial operator.
- t, s - will always be variables reserved as time variables.
- $K(x, y, t)$ - the kernel for the elliptic operator \mathcal{L} as defined in (1.2).
- $\mathcal{K}(t, s)$ - the kernel for \mathcal{D}_t^α as defined in (1.7).
- M_σ^\pm - Pucci's extremal operators (defined in Section 2) for the elliptic spatial operators.
- M_α^\pm - Pucci's extremal operators (defined in Section 2) for the fractional time derivatives.
- λ, Λ - Ellipticity constants as appearing in (1.5) and (1.9).
- $Q_r(x_0, t_0)$ - the space-time cylinder $B_r(x_0) \times (t_0 - r^{2\sigma/\alpha}, t_0)$.
- Q_r - the cylinder centered at the origin $Q_r(0, 0)$.

1.4. Outline. The outline of our paper is as follows: In Section 2 we explain the notion of viscosity solution that will be used in the paper. In Section 3 we use a standard method to show the comparison principle and uniqueness for ordinary differential equations involving \mathcal{D}_t^α . In Section 4 we prove our first main result that solutions to certain ordinary differential equations involving \mathcal{D}_t^α are Hölder continuous. In Section 5 we establish the necessary properties for solutions of (1.10). In Section 6 we prove our second main result that solutions to parabolic equations of type (1.1) are Hölder continuous.

2. VISCOSITY SOLUTIONS AND PUCCI'S EXTREMAL OPERATORS

In order to study weak solutions of equations of type $\mathcal{D}_t^\alpha u - \mathcal{L}u = f$ in nondivergence form we will utilize the notion of viscosity solution. For the elliptic operator \mathcal{L} we recall the notion of Pucci's extremal operators introduced in [5]. For fixed time t we denote the second order difference $\delta(u, x, y, t) := u(x+y, t) + u(x-y, t) - 2u(x, t)$.

We fix two constants $0 < \lambda \leq \Lambda$ and define

$$M_\sigma^+ u(x, t) := \int_{\mathbb{R}^n} \frac{\Lambda \delta(u, x, y, t)_+ - \lambda \delta(u, x, y, t)_-}{|y|^{n+2\sigma}}$$

$$M_\sigma^- u(x, t) := \int_{\mathbb{R}^n} \frac{\lambda \delta(u, x, y, t)_+ - \Lambda \delta(u, x, y, t)_-}{|y|^{n+2\sigma}}.$$

We now define a Pucci-type extremal operator for fractional derivatives of type (1.7). For fixed $x \in \mathbb{R}^n$,

$$M_\alpha^+ u(x, t) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{\Lambda(u(x, t) - u(x, s))_+ - \lambda(u(x, t) - u(x, s))_-}{(t-s)^{1+\alpha}}$$

$$M_\alpha^- u(x, t) := \frac{\alpha}{\Gamma(1-\alpha)} \int_{-\infty}^t \frac{\lambda(u(x, t) - u(x, s))_+ - \Lambda(u(x, t) - u(x, s))_-}{(t-s)^{1+\alpha}}.$$

Since α is reserved for \mathcal{D}_t^α and σ is reserved for the kernel of \mathcal{L} , there should be no confusion between M_α^\pm and M_σ^\pm . These operators give rise to the equations

$$M_\alpha^- u(x, t) - M^+ u(x, t) \leq f(x, t)$$

$$M_\alpha^+ u(x, t) - M^- u(x, t) \geq f(x, t).$$

Since in this paper we show regularity for the parabolic equation when the kernels $\mathcal{K}(t, s)$ satisfy (1.8), we will only consider solutions to

$$\mathcal{D}_t^\alpha u(x, t) - M^+ u(x, t) \leq f(x, t) \quad (2.1)$$

$$\mathcal{D}_t^\alpha u(x, t) - M^- u(x, t) \geq f(x, t). \quad (2.2)$$

When proving regularity for ordinary differential equations we will not assume $\mathcal{K}(t, s)$ satisfies (1.8), so we consider solutions to

$$M_\alpha^- u(t) \leq f(t) \quad (2.3)$$

$$M_\alpha^+ u(t) \geq f(t). \quad (2.4)$$

As in [6] we have the following properties for Pucci's extremal operators.

Proposition 2.1. *For fixed u, v evaluated at fixed (x, t) we have the following properties where M^\pm denote either M_α^\pm or M_σ^\pm .*

- (i) $M^- u \leq M^+ u$.
- (ii) $M^- u = -M^+(-u)$.
- (iii) $M^\pm cu = cM^\pm u$ if $c \geq 0$.
- (iv) $M^+ u + M^- v \leq M^+(u+v) \leq M^+ u + M^+ v$.
- (v) $M^- u + M^- v \leq M^-(u+v) \leq M^- u + M^- v$.

We now define a viscosity solution. We say that an upper semi-continuous function u is a viscosity subsolution of (1.1) (or a solution of (2.1)) if whenever a $C^{2,1}$ function satisfies $\phi \geq u$ on $[t_0 - r, t_0] \times B_\rho(x_0)$ and $\phi(x_0, t_0) = u(x_0, t_0)$ and if v is defined as

$$v(x, t) := \begin{cases} \phi(x, t) & \text{if } (x, t) \in [t_0 - r, t_0] \times B_\rho(x_0) \\ u(x, t) & \text{otherwise,} \end{cases}$$

Then $\mathcal{D}_t^\alpha v(x_0, t_0) - \mathcal{I}v(x_0, t_0) \leq f(x_0, t_0)$ (or v is a solution to (2.1) at (x_0, t_0)). A viscosity supersolution of (1.1) (or a solution of (2.2)) for lower semi-continuous

functions is defined similarly. We point out that a viscosity subsolution (supersolution) of (1.1) is a viscosity solution of (2.1) ((2.2)). A solution is both a subsolution and supersolution, and consequently a continuous function.

The notion of viscosity solutions and supersolutions for $\mathcal{D}_t^\alpha u = f$ (or solutions to (2.3) and (2.4)) are similarly defined. We note that we may extend our class of test functions that touch from above or below to functions ϕ that are C^2 in the x -variable for fixed t and Lipschitz in time for fixed x . It is clear that if function $\mathcal{D}_t^\alpha u$ can be evaluated classically and solves $\mathcal{D}_t^\alpha u = f$ then u is a solution in the viscosity sense. This is made clear in the following two Propositions.

Proposition 2.2. *Let u be a continuous bounded function. Let $\phi \in C^{0,\gamma}$ with $\alpha < \gamma \leq 1$. If $\phi \geq (\leq) u$ on $[t_0 - \epsilon, t_0]$ and $\phi(t_0) = u(t_0)$, then the integral*

$$\int_{-\infty}^{t_0} [u(t_0) - u(s)] \mathcal{K}(t, s) \, ds$$

is well defined and possibly ∞ ($-\infty$) so that $\mathcal{D}_t^\alpha u(t_0)$ is well defined.

Proof. Without loss of generality we assume that $\phi \geq u$ on $[t_0 - \epsilon, t_0]$. Then

$$\begin{aligned} & \int_{-\infty}^{t_0} [u(t_0) - u(s)] \mathcal{K}(t, s) \, ds \\ &= \int_{-\infty}^{t_0 - \epsilon} [u(t_0) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t_0 - \epsilon}^{t_0} [u(t_0) - u(s)] \mathcal{K}(t, s) \, ds \\ &\geq \int_{-\infty}^{t_0 - \epsilon} [u(t_0) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t_0 - \epsilon}^{t_0} [u(t_0) - \phi(s)] \mathcal{K}(t, s) \, ds \\ &= \int_{-\infty}^{t_0 - \epsilon} [u(t_0) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t_0 - \epsilon}^{t_0} [\phi(t_0) - \phi(s)] \mathcal{K}(t, s) \, ds \\ &\geq \int_{-\infty}^{t_0 - \epsilon} [u(t_0) - u(s)] \mathcal{K}(t, s) \, ds - \Lambda \|\phi\|_{C^{0,\gamma}} \int_{t_0 - \epsilon}^{t_0} (t_0 - s)^{\gamma-1-\alpha} \, ds \\ &\geq \int_{-\infty}^{t_0 - \epsilon} [u(t_0) - u(s)] \mathcal{K}(t, s) \, ds - C_1 \\ &\geq -2\Lambda \|u\|_{L^\infty} \int_{-\infty}^{t_0 - \epsilon} (t_0 - s)^{-1-\alpha} \, ds - C_1 \\ &\geq -C_2. \end{aligned}$$

Therefore, the integral is well defined and possibly ∞ . \square

Proposition 2.3. *Let u be a continuous bounded function on $(-\infty, T]$ and assume that for some $t \in (-\infty, T]$ there exists a Lipschitz function touching u by below (above) at t . From Proposition 2.2, the term $\mathcal{D}_t^\alpha u(t)$ is well defined and we have that*

$$\int_{-\infty}^t [u(t) - u(s)] \mathcal{K}(t, s) \, ds \geq (\leq) f(t) \quad (2.5)$$

if and only if $\mathcal{D}_t^\alpha u(t) \leq (\geq) f(t)$ in the viscosity sense.

Proof. Assume the inequality in (2.5). If ϕ touches u from below in $[t - \epsilon, t]$, then

$$\begin{aligned} f(t) &\leq \int_{-\infty}^t [u(t) - u(s)] \mathcal{K}(t, s) \, ds \\ &= \int_{-\infty}^{t-\epsilon} [u(t) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t-\epsilon}^t [u(t) - u(s)] \mathcal{K}(t, s) \, ds \\ &\leq \int_{-\infty}^{t-\epsilon} [u(t) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t-\epsilon}^t [u(t) - \phi(s)] \mathcal{K}(t, s) \, ds \\ &= \int_{-\infty}^{t-\epsilon} [u(t) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t-\epsilon}^t [\phi(t) - \phi(s)] \mathcal{K}(t, s) \, ds. \end{aligned}$$

and so $\mathcal{D}_t^\alpha u \geq f(t)$ in the viscosity sense. Assume now that $\mathcal{D}_t^\alpha u \geq f(t)$ in the viscosity sense. From the assumption, we may touch u from below by a Lipschitz function ϕ in some neighborhood $[t - \epsilon, t]$. We may then find Lipschitz ϕ_k converging uniformly to u in $[t - \epsilon, t]$ with $\phi_k \geq u$ in $[t - \epsilon, t]$. Since the integral in (2.5) is well defined we have from Lebesgue's dominated convergence theorem

$$\begin{aligned} f(t) &\leq \lim_{k \rightarrow \infty} \int_{-\infty}^{t-\epsilon} [u(t) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t-\epsilon}^t [\phi_k(t) - \phi_k(s)] \mathcal{K}(t, s) \, ds \\ &= \lim_{k \rightarrow \infty} \int_{-\infty}^{t-\epsilon} [u(t) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t-\epsilon}^t [u(t) - \phi_k(s)] \mathcal{K}(t, s) \, ds \\ &= \int_{-\infty}^{t-\epsilon} [u(t) - u(s)] \mathcal{K}(t, s) \, ds + \int_{t-\epsilon}^t [u(t) - u(s)] \mathcal{K}(t, s) \, ds \\ &= \int_{-\infty}^t [u(t) - u(s)] \mathcal{K}(t, s) \, ds. \end{aligned}$$

□

The notion of continuity is important for viscosity solutions. If we let

$$\begin{cases} u(t) = t^{\alpha-1} & \text{if } t > 0 \\ u(t) = 0 & \text{if } t \leq 0, \end{cases}$$

then one may explicitly compute that $\partial_t^\alpha u(t) = 0$ for any $t \in \mathbb{R}$. However, u is not upper semi-continuous, and therefore not a viscosity subsolution.

Viscosity solutions are closed under appropriate limits.

Lemma 2.4. *Let u_k be a sequence of continuous bounded viscosity solutions to (2.1) (or (2.2)) in $B_{R_1} \times [-R_2, 0]$, iconverging in $\mathbb{R}^n \times (-\infty, T]$ to u_0 bounded and continuous. Then u_0 is a viscosity solution to (2.1) (or (2.2)) in $B_{R_1} \times [-R_2, 0]$.*

Proof. The proof is standard and straightforward from the definition of viscosity solutions (see [6]). □

3. APPROXIMATING SOLUTIONS

In order to show uniqueness for an ordinary differential equation such as $\mathcal{D}_t^\alpha u = f$ we use the notion of sup- and inf-convolution. For a bounded and upper-semicontinuous function u on $(-\infty, t_0]$, for $t \leq t_0$ we define

$$u^\epsilon(t) := \sup_{s \leq t} \left\{ u(s) + \frac{1}{\epsilon}(s - t) + \epsilon \right\}. \quad (3.1)$$

If u is bounded and lower-semicontinuous on $(-\infty, t_0]$, for $t \leq t_0$ we define

$$u_\epsilon(t) := \inf_{s \leq t} \left\{ u(s) - \frac{1}{\epsilon}(s - t) - \epsilon \right\}. \quad (3.2)$$

We have the following properties

Proposition 3.1. *For u^ϵ as defined in (3.1) and $t \leq t_0$ the following hold*

- (1) *there exists $t^* \leq t$ such that $u^\epsilon(t) = u(t^*) + \frac{1}{\epsilon}(t^* - t) + \epsilon$.*
- (2) *$u^\epsilon(t) \geq u(t) + \epsilon$.*
- (3) *$u^\epsilon(t_2) - u^\epsilon(t_1) \geq \epsilon^{-1}(t_2 - t_1)$ for $t_1 < t_2$.*
- (4) *$0 < \epsilon_1 < \epsilon_2 \Rightarrow u^{\epsilon_1}(t) \leq u^{\epsilon_2}(t)$.*
- (5) *$t - t^* \leq 2\epsilon \sup |u|$.*
- (6) *$0 < u^\epsilon(t) - u(t) \leq u(t^*) - u(t) + \epsilon$.*

Proof. All properties except for (3) are as in Lemma 5.2 in [6]. For property (3) we note that for $t \leq t_1 \leq t_2$ we have

$$\begin{aligned} u^\epsilon(t_2) &\geq u(t) + \frac{1}{\epsilon}(t - t_2) + \epsilon \\ &= u(t) + \frac{1}{\epsilon}(t - t_1) + \epsilon + \frac{1}{\epsilon}(t_1 - t_2). \end{aligned}$$

Taking the supremum over $t \leq t_1$ we obtain

$$u^\epsilon(t_2) - u^\epsilon(t_1) \geq -\epsilon^{-1}(t_2 - t_1).$$

□

Using the properties listed in Proposition 3.1, it is standard to show the following Proposition which is analogous to Theorem 5.1 in [6] and Propositions 5.4 and 5.5 in [5].

Proposition 3.2. *If u is bounded and lower-semicontinuous (upper-semicontinuous) in $(-\infty, T]$, then $u_\epsilon(-u^\epsilon)$ Γ -converges to $u(-u)$. If u is continuous then u^ϵ, u_ϵ converge uniformly to u . Furthermore, if $\mathcal{D}_t^\alpha u \geq (\leq) f$, then there exists $d_\epsilon \rightarrow 0$ as $\epsilon \rightarrow 0$ such that $\mathcal{D}_t^\alpha u_\epsilon \geq f - d_\epsilon$ ($\mathcal{D}_t^\alpha u^\epsilon \leq f + d_\epsilon$).*

Lemma 3.3. *Let u be bounded and upper-semicontinuous and v be bounded and lower semi-continuous on $(-\infty, T]$. Let f, g be continuous functions. If $\mathcal{D}_t^\alpha u \leq f$ and $\mathcal{D}_t^\alpha v \geq g$ in the viscosity sense, then $\mathcal{D}_t^\alpha(v - u) \geq g - f$ in the viscosity sense.*

Proof. We take the approximating solutions u^ϵ, v_ϵ with $\mathcal{D}_t^\alpha u^\epsilon \leq f + d_\epsilon$ and $\mathcal{D}_t^\alpha v_\epsilon \geq f - d_\epsilon$. From property (3) in Proposition 3.1 we have that at every point u^ϵ can be touched from above by a Lipschitz function and v_ϵ can be touched from below. Then also $v_\epsilon - u^\epsilon$ can be touched from below by a Lipschitz function at every point. From Propositions 2.2 and 2.3, the terms $\mathcal{D}_t^\alpha u^\epsilon, \mathcal{D}_t^\alpha v_\epsilon, \mathcal{D}_t^\alpha(v_\epsilon - u^\epsilon)$ are well defined and at any point t

$$\mathcal{D}_t^\alpha(v_\epsilon - u^\epsilon(t)) = \mathcal{D}_t^\alpha v_\epsilon(t) - \mathcal{D}_t^\alpha u^\epsilon(t) \geq g - f - 2d_\epsilon.$$

Then from Proposition 2.3 we conclude that $\mathcal{D}_t^\alpha(v_\epsilon - u^\epsilon) \geq g - f - 2d_\epsilon$. Letting $\epsilon \rightarrow 0$ we obtain from Lemma 2.4 that $\mathcal{D}_t^\alpha(v - u) \geq g - f$ in the viscosity sense. □

We are now able to prove a comparison principle.

Theorem 3.4. *Let u be bounded and upper-semicontinuous and v be bounded and lower semi-continuous on $(-\infty, T_2]$. Let f be a continuous function and assume that $\mathcal{D}_t^\alpha u \leq f \leq \mathcal{D}_t^\alpha v$ on $(T_1, T_2]$ with $u \leq v$ on $(-\infty, T_1]$. Then $u \leq v$ on $(-\infty, T_2]$, and if $u(t_0) = v(t_0)$ for some $t_0 \in (T_1, T_2]$, then $u(t) = v(t)$ for all $t \leq t_0$.*

Proof. Suppose there exists $t_1 \in (T_1, T_2]$ such that $v(t_1) - u(t_1) \leq 0$. Let $v - u$ achieve its minimum in $[T_1, T_2]$ at $t_0 \in (T_1, T_2]$. From Lemma 3.3, we have $\mathcal{D}_t^\alpha(v - u) \geq 0$ in the viscosity sense. Since $v - u \geq 0$ for $t \leq T_1$, then $(v - u)$ is touched from below by the constant function $v(t_0) - u(t_0)$ at t_0 . From Propositions 2.2 and 2.3 the term $\mathcal{D}_t^\alpha(v - u)(t_0)$ is well defined and

$$\int_{-\infty}^{t_0} [(v - u)(t_0) - (v - u)(s)] \mathcal{K}(t, s) \, ds \geq 0.$$

Since $(v - u)$ achieves a minimum over $(-\infty, t_0]$ at t_0 we also have

$$0 \geq \int_{-\infty}^{t_0} [(v - u)(t_0) - (v - u)(s)] \mathcal{K}(t, s) \, ds.$$

Then if $(v - u)(t_0) \leq 0$ we have that $(v - u)(t) = 0$ for all $t \leq t_0$. Then either $(v - u)(t_0) > 0$ or $(v - u)(t) = 0$ for all $t \leq t_0$. \square

With a comparison principle available, we may use Perron's method to prove existence of solutions.

Theorem 3.5. *Let $\phi(t)$ be continuous on $(-\infty, T_1]$ and f continuous on $[T_1, T_2]$. There exists a unique viscosity solution u to*

$$\begin{cases} \mathcal{D}_t^\alpha u(t) = f(t) & \text{for } t \in [T_1, T_2] \\ u(t) = \phi(t) & \text{if } t \leq T_1 \end{cases}$$

on $(T_1, T_2]$.

The proof of Theorem 3.5 is standard. We forgo the proof since the ideas and methods are later used in the proof of Lemma 5.2. One may also show existence for solutions of (1.1); however, since the main focus of this paper is the regularity of solutions we do not include the result here.

Later in the paper we will need that the continuous divergence solutions constructed in [2, 3] are also viscosity solutions.

Theorem 3.6. *Let f be a continuous function on $[-2, 0]$ with $f(-2) = 0$. Assume the kernel $\mathcal{K}(t, s)$ of \mathcal{D}_t^α satisfies (1.8) and (1.9). Then the weak divergence solution v constructed in [2] of*

$$\begin{cases} \mathcal{D}_t^\alpha v = f & \text{in } (-\infty, 0]. \\ v = 0 & \text{for } t \leq -2, \end{cases} \quad (3.3)$$

is also a viscosity solution and hence the unique viscosity solution.

Proof. In [2, 3] the solution v is constructed with an ϵ approximation. Via recursion we find the solution v_ϵ to

$$\epsilon \sum_{i < j} [v_\epsilon(\epsilon j) - v_\epsilon(\epsilon i)] \mathcal{K}(\epsilon j, \epsilon i) = f(\epsilon j), \quad (3.4)$$

where $i, j \in \mathbb{Z}$. From [2, 3], $v_\epsilon \rightarrow v$ where v solves the divergence form equation

$$\begin{aligned} & \int_{-\infty}^T \int_{-\infty}^t [v(t) - v(s)][\phi(t) - \phi(s)] \mathcal{K}(t, s) \, ds \, dt \\ & + \int_{-\infty}^T \int_{-\infty}^{2t-T} v(t) \phi(t) \mathcal{K}(t, s) \, ds \, dt - \int_{-\infty}^T v(t) \mathcal{D}_t^\alpha \phi(t) \, dt \\ & = \int_{-\infty}^T f(t) \phi(t) \, dt, \end{aligned} \quad (3.5)$$

for all $T \leq 0$ and ϕ bounded and Lipschitz on $(-\infty, 0]$. We extend v_ϵ to all of $(-\infty, 0]$ by $v_\epsilon(t) = v(\epsilon i)$ where $\epsilon(i-1) < t \leq \epsilon j$. We now show that v is a viscosity solution. The main idea in the following computations is that since (3.4) is a discrete equation, then v_ϵ is also a viscosity-type discretized solution, and so the limit will also be a viscosity solution. We now let ψ be a Lipschitz function touching v strictly from above at $t_0 \leq 0$, that is $v(t) < \psi(t)$ for $t < t_0$ and $v(t_0) = \psi(t_0)$. We let δ_ϵ be such that $\psi + \delta_\epsilon$ touches v_ϵ from above at $t_\epsilon \leq t_0$. Since ψ touches v strictly from above at t_0 , then $t_\epsilon \rightarrow t_0$ as $\epsilon \rightarrow 0$. Now If $\epsilon(j_\epsilon - 1) < t_\epsilon \leq \epsilon j_\epsilon$, then since $\psi + \delta_\epsilon$ touches v_ϵ from above and (3.4) is a discrete equation, we have that

$$\epsilon \sum_{i < j_\epsilon} [\psi(t_\epsilon) - \psi(\epsilon i)] \mathcal{K}(\epsilon j_\epsilon, \epsilon i) \geq f(\epsilon j_\epsilon).$$

Now

$$\begin{aligned} \epsilon \sum_{i < j_\epsilon} [\psi(t_\epsilon) - \psi(\epsilon i)] \mathcal{K}(\epsilon j_\epsilon, \epsilon i) &= \epsilon \sum_{i < j_\epsilon} [\psi(t_\epsilon) - \psi(\epsilon j_\epsilon)] \mathcal{K}(\epsilon j_\epsilon, \epsilon i) \\ &\quad + \epsilon \sum_{i < j_\epsilon} [\psi(\epsilon j_\epsilon) - \psi(\epsilon i)] \mathcal{K}(\epsilon j_\epsilon, \epsilon i) \\ &= (I) + (II). \end{aligned}$$

Since ψ is Lipschitz we have

$$\begin{aligned} |(I)| &= \left| \epsilon \sum_{i < j_\epsilon} [\psi(t_\epsilon) - \psi(\epsilon j_\epsilon)] \mathcal{K}(\epsilon j_\epsilon, \epsilon i) \right| \\ &\leq \epsilon \sum_{i < j_\epsilon} [C\epsilon] \mathcal{K}(\epsilon j_\epsilon, \epsilon i) \\ &\leq \epsilon^2 \sum_{i < j_\epsilon} \Lambda \epsilon^{-1-\alpha} (j_\epsilon - i)^{-1-\alpha} \\ &= \epsilon^{1-\alpha} \Lambda \sum_{i=1}^{\infty} i^{-1-\alpha}. \end{aligned}$$

Then $|(I)| \rightarrow 0$ as $\epsilon \rightarrow 0$. Since ψ is Lipschitz continuous it follow that $(II) \rightarrow \mathcal{D}_t^\alpha \psi(t_0)$ as $\epsilon \rightarrow 0$. Since f is continuous and $t_\epsilon \rightarrow t_0$ as $\epsilon \rightarrow 0$, we have that $f(\epsilon j_\epsilon) \rightarrow f(t_0)$. Then $\mathcal{D}_t^\alpha \psi(t_0) \geq f(t_0)$. The proof if ψ touches from below is similar. Then v is a viscosity solution. \square

This next Corollary will be useful when we want to show a limit is not equivalently zero.

Corollary 3.7. *Let v be the viscosity solution to (3.3) as in Theorem 3.6. Assume also that $v \geq 0$. Then for $-2 < t \leq 0$ we have*

$$\frac{\lambda\alpha}{\Gamma(1-\alpha)} \int_{-2}^t \frac{v(s)}{(t-s)^\alpha} ds \leq \int_{-2}^t f(s) ds \leq \frac{\Lambda\alpha}{\Gamma(1-\alpha)} \int_{-2}^t \frac{v(s)}{(t-s)^\alpha} ds. \quad (3.6)$$

Proof. We take $\phi \equiv 1$ in (3.5). Then

$$\begin{aligned} \int_{-2}^T f(t) dt &= \int_{-\infty}^T \int_{-\infty}^{2t-T} v(t) K(t, s) ds dt \\ &\leq \frac{\alpha\Lambda}{\Gamma(1-\alpha)} \int_{-\infty}^{2t-T} \int_{-\infty}^t \frac{v(t)}{(t-s)^{1+\alpha}} ds dt \\ &= \frac{\alpha\Lambda}{\Gamma(1-\alpha)} \int_{-2}^T \frac{v(t)}{(T-t)^\alpha} dt. \end{aligned}$$

The bound from the other side is shown in the same manner. \square

4. HÖLDER CONTINUITY FOR THE TIME DERIVATIVE

We will follow the ideas in [5] to prove the Hölder continuity. The outline and statements of the Lemmas are intentionally similar to those in Sections 8-12 in [5] so that the reader may compare and contrast properties of the operator \mathcal{D}_t^α with properties of \mathcal{L} . Rather than work with the concave envelope we will work with a different envelope.

$$\mathcal{M}_u(t) := \sup_{s \leq t} u(s).$$

Lemma 4.1. *Assume $M_\alpha^- u(t) \leq f(t)$ and $\mathcal{M}_u(t_0) = u(t_0)$. Let $r_k := 2^{-1/(1-\alpha)} 2^{-k}$ and $R_k(t) := [t - r_k, t - r_{k+1}]$. There is a constant C_0 depending on λ, Λ but not α such that for any $M > 0$, there exists k such that*

$$|R_k(t_0) \cap \{u(s) < u(t_0) - Mr_k\}| \leq C_0 \frac{f(t_0)}{M} |R_k(t_0)|. \quad (4.1)$$

Proof. We first note that $|R_k(t_0)| = r_{k+1} = r_k/2$. Since $\mathcal{M}_u(t_0) = u(t_0)$, then u is touched from above by the constant $\mathcal{M}_u(t_0)$. Then from Proposition 2.3 we have

$$\int_{-\infty}^{t_0} [u(t_0) - u(s)] K(t_0, s) ds \leq f(t_0).$$

Suppose that the conclusion is not true. It then follows that

$$\begin{aligned} f(t_0) &\geq M_\alpha^- u(t_0) \geq \frac{\alpha\lambda}{\Gamma(1-\alpha)} \int_{-\infty}^{t_0} \frac{u(t_0) - u(s)}{(t_0 - s)^{1+\alpha}} ds \\ &\geq \frac{\alpha\lambda}{\Gamma(1-\alpha)} \int_{t_0 - 2^{-1/(1-\alpha)}}^{t_0} \frac{u(t_0) - u(s)}{(t_0 - s)^{1+\alpha}} ds \\ &\geq \frac{\alpha\lambda}{\Gamma(1-\alpha)} \sum_{k=0}^{\infty} \int_{R_k(t_0)} \frac{u(t_0) - u(s)}{r_{k+1}^{1+\alpha}} ds \\ &\geq \frac{\alpha\lambda}{\Gamma(1-\alpha)} Mr_k \frac{C_0 f(t_0)}{M} \frac{|R_k(t_0)|}{r_{k+1}^{1+\alpha}} \\ &= 2^\alpha \frac{\alpha\lambda}{\Gamma(1-\alpha)} C_0 f(t_0) \sum_{k=0}^{\infty} r_k^{1-\alpha}. \end{aligned}$$

Now

$$\sum_{k=0}^{\infty} r_k^{1-\alpha} = \frac{1}{2} \sum_{k=0}^{\infty} \left(\frac{1}{2^{1-\alpha}} \right)^k = \frac{1}{2} \left[\frac{1}{1 - 2^{-(1-\alpha)}} \right].$$

Then

$$f(t_0) \geq \frac{\alpha \lambda}{\Gamma(1-\alpha)} \frac{C_0 f(t_0)}{2^{1-\alpha} - 1}.$$

From the following relation on the Gamma function $\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin(\pi z)}$ it follows that for $\alpha_0 < \alpha \leq 1$, the right hand side of the above equation is bounded below. Then for C_0 large enough we obtain a contradiction. \square

Remark 4.2. In the above proof it is clear that by choosing C_0 larger (say for instance $2C_0$) that for fixed α and M bounded from above we may choose $k \leq N$ for some large N . This N will depend on α , but this gives a bound from below on r_k such that (4.1) holds.

Corollary 4.3. *For any $\epsilon_0 > 0$, there exists C such that for u as defined in Lemma 4.1, there exists $r \in (0, 2^{-1/(1-\alpha)})$ such that if $\mathcal{M}_u(t) = u(t)$ and $M_\alpha^- u(t) \leq f(t)$, then there exists a constant c_α (depending on α) and $r \in (c_\alpha, 2^{-1/(1-\alpha)})$ such that*

$$\frac{|\{s \in [t-r, t-r/2] : u(s) < u(t) - Cf(t)r\}|}{|[t-r, t-r/2]|} \leq \epsilon_0 \quad (4.2)$$

Proof. By choosing $M = Cf(t)\epsilon_0^{-1}$ we obtain (4.2) from Lemma 4.1 and Remark 4.2. \square

Lemma 4.4. *Let $u(t) \leq 0$ for $t < -2$ and $M_\alpha^- u \leq f$ on $[-2, T]$. There exists a constant C depending on λ, α_0 and there exists finitely many intervals I_j with disjoint interiors and with length between c_α and $2^{-(1-\alpha)}$ such that*

$$\sup u^+ \leq C \sum_j \max_{I_j} f^+(t) |I_j|. \quad (4.3)$$

Furthermore, if $I_j = [t_j - r_j, t_j]$ is such an interval, and $[t_j - 2r_j, t_j] \subset [-2, T]$, then

$$|\{s \in [t_j - 2r_j, t_j - r_j] : u(s) \geq u(t_j) - Cr_j\}| \geq \mu r_j. \quad (4.4)$$

Remark 4.5. Notice that as $\alpha \rightarrow 1$ the inequality (4.3) becomes

$$\sup u \leq C \int_{\{u=\mathcal{M}_u\}} f^+(s) ds.$$

Proof. We will choose the finite sequence of intervals inductively. Let u^+ achieve its maximum on $[-2, T]$ at the point t_1 . From Corollary 4.3 there exists r_1 with $c_\alpha \leq r_1 \leq 2^{-1-\alpha}$ such that (4.4) holds for $I_1 = [t_1 - r_1, t_1]$.

$$\mathcal{M}_{u^+}(t_1) - \mathcal{M}_{u^+}(t_1 - r_1) \leq u(t_1) - u(s) \text{ for any } s \in [t_1 - 2r_1, t_1 - r_1].$$

Since $[t_1 - 2r_1, t_1 - r_1]$ satisfies (4.4), we then have that

$$\mathcal{M}_{u^+}(t_1) - \mathcal{M}_{u^+}(t_1 - r_1) \leq Cf(t_1)r_1 = Cf(t_1)|I_1|.$$

Proceeding inductively, supposing I_{j-1} has been chosen, we choose

$$t_j := \max\{-2 \leq s \leq t_{j-1} - r_{j-1} : u(s) = \mathcal{M}_u(s)\}.$$

If no such t_j exists then the process terminates. Otherwise, by Corollary 4.3 we choose $r_j \in (c_\alpha, 2^{-1/(1-\alpha)})$ such that $[t_j - 2r_j, t_j - r_j]$ satisfies (4.4). We label $I_j = [t_j - r_j, t_j]$. As in the case of I_1 we obtain

$$\mathcal{M}_{u^+}(t_j) - \mathcal{M}_{u^+}(t_j - r_j) \leq C f(t_j) |I_j|.$$

Since each $r > c_\alpha$ the process will eventually terminate. Now

$$\{u = \mathcal{M}_{u^+}\} \subseteq \cup_j I_j,$$

and so

$$\sup u^+ \leq \sum_j |\mathcal{M}_{u^+}(t_j) - \mathcal{M}_{u^+}(t_j - r_j)| \leq C \sum_j \max_{I_j} f^+(t) |I_j|.$$

Finally, from how each I_j is chosen we have (4.4) holds for each $[t_j - 2r_j, t_j - r_j]$ and also $\overset{\circ}{I}_j \cap \overset{\circ}{I}_k = \emptyset$ for $j \neq k$. \square

For this next Lemma for an interval $I_j = [t_j - r_j, t_j]$ we define

$$2I_j := [t_j - 2r_j, t_j].$$

Lemma 4.6. *Let $A \subset [0, 1]$ and let c_1, c_2 be two positive constants. Let $\{I_j : j \in \mathcal{I}\}$ be a collection of closed intervals with nonempty interior satisfying the following for every $j \in \mathcal{I}$:*

- (i) $2I_j \subset [0, 1]$
- (ii) $\overset{\circ}{I}_i \cap \overset{\circ}{I}_j = \emptyset$ for every $i \neq j$
- (iii) $\sum_{j \in \mathcal{I}} |I_j| > c_1$
- (iv) $|A \cap 2I_j| \geq c_2 |2I_j|$,

then $|A| \geq (c_1 \cdot c_2)/3$.

Proof. We first assume the collection $\{I_j\}$ is finite. We will choose a subset $\mathcal{J} \subseteq \mathcal{I}$. Recall that each interval has nonempty interior and the intersection of the interiors is empty. We will pick a subcollection of intervals as follows: We choose and label I_1 as the interval with the farthest right end point. If there is another interval I_k for some $k \in \mathcal{I}$ such that

- (1) $2I_k \cap 2I_1 \neq \emptyset$ and
- (2) $2I_k \setminus (2I_k \cap 2I_1) \neq \emptyset$,

then we choose I_2 to be the interval satisfying (1) and (2) such that $2I_k$ has the farthest left endpoint. If no $2I_k$ satisfies (1) and (2) then we choose I_2 to be the interval whose right end point is closest to the left end point of $2I_1$. We then choose all remaining intervals in the same manner. It is clear from the construction that each interval from the collection $\{2I_j\}$ for $j \in \mathcal{J}$ can intersect at most two other intervals from the same collection. It is also clear that

$$\bigcup_{i \in \mathcal{I}} I_i \subset \bigcup_{j \in \mathcal{J}} 2I_j, \tag{4.5}$$

Then

$$\begin{aligned}
|A| &\geq \frac{1}{2} \sum_{j \in \mathcal{J}} |A \cap 2I_j| \\
&\geq \frac{c_2}{3} \sum_{j \in \mathcal{J}} |2I_j| \\
&\geq \frac{c_2}{3} \sum_{i \in \mathcal{I}} |I_i| \\
&\geq \frac{c_1 c_2}{3}.
\end{aligned}$$

If the collection $\{I_i\}$ is infinite, then we simply choose a finite subcollection of intervals whose measure is within ϵ , and then let $\epsilon \rightarrow 0$. \square

We now prove the most important Lemma.

Lemma 4.7. *Let $0 < \alpha_0 \leq \alpha \leq 1$. There exists $\epsilon_0 > 0$, $0 < \mu < 1$, and $M > 1$ all depending on $\alpha_0, \lambda, \Lambda$ such that if*

- (i) $u \geq 0$ in $(-\infty, 1]$
- (ii) $u(1) \leq 1/2$
- (iii) $M_\alpha^+ u \geq -\epsilon_0$ in $[-1, 1]$,

then

$$|\{u \leq M\} \cap [0, 1]| > \mu.$$

Proof. We utilize the function

$$\Phi(t) := \begin{cases} 0 & \text{for } t \leq 3/4 \\ 4(t - 3/4) & \text{for } t \geq 3/4. \end{cases}$$

Notice that $M_\alpha^- u \leq 4\lambda$ for all $0 < \alpha \leq 1$. We now apply Lemma 4.4 to $v := \Phi - u$. Since $u(1) \leq 1/2$ we have that $v(1) \geq 1/2$. We have From estimate (4.3) we have

$$\begin{aligned}
1/2 \leq \sup v &\leq C \sum_j \max_{I_j} M_\alpha^-(\Phi - u) |I_j| && \text{from (4.3)} \\
&\leq C(\epsilon_0 + 4\lambda C_1) \sum_j |I_j| && \text{from Proposition 2.1} \\
&\leq C\epsilon_0 + 4\lambda C_1 \sum_j |I_j|.
\end{aligned}$$

Then for ϵ_0 small enough we obtain

$$\frac{1}{4} \leq C \sum_j |I_j|.$$

Now Φ is supported in $[3/4, 1]$. Then if $I_j = [t_j - r_j/4, t_j]$, then $t_j \in [3/4, 1]$ and so $2I_j := [t_j - r_j, t_j] \subset [0, 1]$. From (4.4)

$$|\{s \in 2I_j : v(s) \geq v(t_j) - Cr_j\}| \geq \mu |I_j|.$$

We recall that $r_j \leq 2^{-(1-\alpha)}$ and $v(t_j) \geq 0$, so that

$$|\{s \in 2I_j : \Phi(s) - u(s) \geq -C\}| \geq \mu |I_j|.$$

Furthermore, $\Phi(s) \leq 1$, and so

$$|\{s \in 2I_j : 1 + C \geq u(s)\}| \geq \mu|I_j|.$$

We now use Lemma 4.6 to conclude that

$$|\{s \in [0, 1] : 1 + C \geq u(s)\}| \geq \mu_1.$$

□

Using Lemma 4.7 with Lemma 4.2 from [6] one proves the following two Lemmas just as in [6].

Lemma 4.8. *Let u be as in Lemma 4.7, then*

$$|\{u > M^k\} \cap [0, 1]| \leq (1 - \mu)^k.$$

Lemma 4.9. *Let u be as in Lemma 4.7, then*

$$|\{u > t\} \cap [0, 1]| \leq dt^{-\epsilon}$$

where $d, \epsilon > 0$ depend only on $\lambda, \Lambda, \alpha_0$.

We now have

Theorem 4.10. *Let $u \geq 0$ in $(-\infty, 0]$ and $M_\alpha^+ u \geq -C_0$ in $[-2r, 0]$. Assume $\alpha \geq \alpha_0 > 0$. Then*

$$|\{u > t\} \cap [-r, 0]| \leq dr(2u(0) + C_0 r^\alpha \epsilon_0^{-1})^\epsilon t^{-\epsilon} \text{ for every } t$$

where the constants d, ϵ depend only on λ, Λ , and α_0 as in Lemma 4.9.

Proof. We rescale and let

$$v(t) := \frac{u(rt)}{2u(0) + C_0 r^\alpha \epsilon_0^{-1}}.$$

Then $M_\alpha^+ v \geq -\epsilon_0$ on $[-2, 0]$ and $v(0) \leq 1/2$, so we may apply Lemma 4.9 to v and conclude

$$|\{v > t\} \cap [-1, 0]| \leq dt^{-\epsilon},$$

which written in terms of u becomes

$$|\{u > t\} \cap [-r, 0]| \leq dr[2u(0) + C_0 r^\alpha \epsilon_0^{-1}]^\epsilon t^{-\epsilon}.$$

□

Theorem 1.1 will follow from the following

Theorem 4.11. *Let $\alpha_0 < \alpha < 1$ for some $\alpha_0 > 0$. Let u be a bounded in $(-\infty, 0]$ and continuous on $[-1, 0]$. If*

$$M_\alpha^+ u \geq -C_0 \text{ in } (-1, 0],$$

$$M_\alpha^- u \leq C_0 \text{ in } (-1, 0],$$

then there is a $\beta > 0$ and $C > 0$ both depending only on $\lambda, \Lambda, \alpha_0$ such that $u \in C^{0, \beta}([-1/2, 0])$ and

$$\|u\|_{C^{0, \beta}([-1/2, 0])} \leq C(\|u\|_{L^\infty} + C_0).$$

Theorem 4.11 follows from Theorem 4.10 and the following Lemma

Lemma 4.12. *Let $-1/2 \leq u \leq 1/2$ in $(-\infty, 0]$. There exists δ_0 depending only on λ, Λ and $0 < \alpha_0 < 1$ such that if*

$$\begin{aligned} M_\alpha^+ u &\geq -\delta_0 & \text{in } [-1, 0] \\ M_\alpha^- u &\leq \delta_0 & \text{in } [-1, 0], \end{aligned}$$

then there is a β depending only on λ, Λ , and α_0 such that if $1 > \alpha \geq \alpha_0$

$$|u(t) - u(0)| \leq C|t|^\beta \quad (4.6)$$

For some constant C .

Proof. The proof is nearly the same as in [5]. We give the details. We will construct a sequence m_k, M_k with $m_k \leq u \leq M_k$ and

$$m_k \leq u \leq M_k \text{ in } [-4^{-k}, 0] \quad (4.7)$$

$$m_k \leq m_{k+1} \text{ and } M_k \geq M_{k+1} \quad (4.8)$$

$$M_k - m_k = 4^{-\beta k}. \quad (4.9)$$

This will show (4.6) with constant $C = 2^\beta$. For $k = 0$ we choose $m_0 = -1/2$ and $M_0 = 1/2$ and by assumption we have (4.7). We proceed by induction. Assume (4.7) holds up to k . We have either

$$\left| \{u \geq (M_k + m_k)/2\} \cap [-3 \cdot 4^{-(k+3)}, -4^{-(k+2)}] \right| \geq 4^{-(k+4)}$$

or

$$\left| \{u \leq (M_k + m_k)/2\} \cap [-3 \cdot 4^{-(k+3)}, -4^{-(k+2)}] \right| \geq 4^{-(k+4)}.$$

If we assume the first then we define

$$v(t) := \frac{u(4^{-k}t) - m_k}{(M_k - m_k)/2}.$$

We have that $v \geq 0$ in $[-1, 0]$ and $|\{v \geq 1\} \cap [-1/2, -1/4]| \geq 1/16$. We also have that

$$M_\alpha^+ v \geq \frac{-4^{-k\alpha}\delta_0}{(M_k - m_k)/2} = -2\delta_0 4^{k(\beta-\alpha)} \geq -2\delta_0$$

as long as $\beta < \alpha$. From the inductive hypothesis, for any $0 \leq j < k$ we have

$$v \geq \frac{m_{k-j} - m_k}{2(M_k - m_k)} \geq \frac{m_{k-j} - M_{k-j} + M_k - m_k}{2(M_k - m_k)} \geq 2(1 - 4^{\beta j}),$$

and so $v \geq -2(|4t|^\beta - 1)$ outside $[-1, 0]$. We define $w := \max\{v, 0\}$. For any $t \in [-3/4, 0]$ we have

$$M_\alpha^+ w(t) - M_\alpha^+ v(t) = \frac{\alpha}{\Gamma(1-\alpha)} \int_{\{v < 0\} \cap \{t < -1\}} \frac{\Lambda v(s)}{(t-s)^{1+\alpha}} \quad (4.10)$$

$$\geq \frac{\alpha\Lambda}{\Gamma(1-\alpha)} \int_{-\infty}^{-1} \frac{-2(|4s|^\beta - 1)}{(t-s)^{1+\alpha}}. \quad (4.11)$$

Thus $M_\alpha^+ w - M_\alpha^+ v \geq -2\delta_0$ for β small enough. We then have $M_\alpha^+ w \geq -4\delta_0$, and so for any $t_0 \in [-1/4, 0]$ we may apply Theorem 4.10 to w to obtain

$$\begin{aligned} \frac{1}{16} &\leq |\{w > 1\} \cap [-3/8, -1/4]| \leq |\{w > 1\} \cap [-3/8, t_0]| \\ &\leq d(1/8)(2u(t_0) + 4(1/8)^\alpha \delta_0 e_0^{-1})^\epsilon. \end{aligned}$$

Thus, for δ_0 small enough we have $u(t_0) \geq \theta > 0$ for any $t_0 \in [-1/4, 0]$. We let $M_{k+1} = M_k$ and $m_{k+1} = m_k + \theta(M_k - m_k)/2$. Then, $M_{k+1} - m_{k+1} = (1 - \theta/2)4^{-\beta k}$. Choosing β and θ small with $(1 - \theta/2) = 4^{-\beta}$ we have $M_{k+1} - m_{k+1} = 4^{-\beta(k+1)}$.

If on the other hand $|\{u \leq (M_k + m_k)/2 \cap [-3 \cdot 4^{-(k+3)}, -4^{-(k+2)}]\}| \geq 4^{-(k+4)}$, then we define

$$v := \frac{M_k - u(4^{-k}t)}{(M_k - m_k)/2}$$

and use that $M_\alpha^- u \leq \delta_0$. \square

5. A BARRIER

In this section we construct a subsolution to an ordinary differential equation that will allow us to prove the Hölder continuity. We begin with the following

Lemma 5.1. *Let $u \in C^{0,\beta}((-\infty, 0])$ with $\alpha < \beta < 1$. Then $M_\alpha^\pm u$ is continuous on $(-\infty, 0]$.*

Proof. We will show that for fixed t and any $\epsilon > 0$, there exists h_0 such that if $0 \leq h < h_0$, then

$$|M_\alpha^\pm u(t+h) - M_\alpha^\pm u(t)| < \epsilon. \quad (5.1)$$

For fixed $\xi < t$ and for $0 \leq h$, we have

$$\begin{aligned} & \left| \frac{\alpha}{\Gamma(1-\alpha)} \int_\xi^{t+h} \frac{\Lambda[u(t+h) - u(s)]_+ - \lambda[u(t+h) - u(s)]_-}{(t+h-s)^{1+\alpha}} ds \right| \\ & \leq \frac{\alpha}{\Gamma(1-\alpha)} \Lambda \|u\|_{C^{0,\beta}} \frac{(t+h-\xi)^{\beta-\alpha}}{\beta-\alpha}. \end{aligned}$$

We may choose ξ close enough to t and choose h small enough, so that the above inequality is less than $\epsilon/2$. Now since u is continuous, if $t > \xi$, then both

$$\left| \int_{-\infty}^\xi \frac{[u(t+h) - u(s)]_-}{(t+h-s)^{1+\alpha}} - \frac{[u(t) - u(s)]_-}{(t-s)^{1+\alpha}} ds \right| \rightarrow 0$$

and

$$\left| \int_{-\infty}^\xi \frac{[u(t+h) - u(s)]_+}{(t+h-s)^{1+\alpha}} - \frac{[u(t) - u(s)]_+}{(t-s)^{1+\alpha}} ds \right| \rightarrow 0$$

as $h \rightarrow 0$. Then we may choose h small enough, so that (5.1) holds. \square

For the next Lemma we will require a solution to an ordinary differential equation. We recall that from Theorem 3.5 if f is continuous on $[-2, 0]$ with $f(-2) = 0$. Then exists a viscosity solution u to the differential equation

$$\begin{cases} u(t) = 0 & \text{for } t \leq -2 \\ \mathcal{D}_t^\alpha u = f(t) & \text{on } (-\infty, 0]. \end{cases}$$

That u is a viscosity solution on $(-2, 0]$ is a direct result from Theorem 3.5. Since $f(t) = u(t) = 0$ for $t \leq -2$ it is immediate that u is also a solution on $(-\infty, -2]$ as well. From Theorem 1.1 the solution u is Hölder continuous.

In the next Lemma we will also utilize the following bump function. Let $\eta \geq 0$ with support in $[0, 1]$. Let $\eta' \geq 0$ for $t \leq 1/2$ and $\eta' \leq 0$ for $t \geq 1/2$. We note that $\mathcal{D}_t^\alpha \eta \leq C$ for some C independent of α and $\mathcal{D}_t^\alpha \eta(t) < 0$ for $t > 1$. We will use

$$\eta_\epsilon := \epsilon \eta(t/\epsilon).$$

Lemma 5.2. *Let f be smooth on $[-2, 0]$ with $f(-2) = 0$. Let u be the viscosity solution to*

$$\begin{cases} \mathcal{D}_t^\alpha u(t) = f & \text{on } (-\infty, 0] \\ u(t) = 0 & \text{for } t \leq -2. \end{cases} \quad (5.2)$$

Then there exists a sequence of Lipschitz subsolutions $\{u_k\}$ to the above equation with $u_k \leq u$ and $u_k \rightarrow u$ uniformly on $[-2, 0]$.

Proof. We consider for fixed $M > \sup f$ the set

$$\mathfrak{K}_1 := \{w : -M \leq D_t^\alpha w \leq f \text{ on } [0, 2], \text{ and } w \text{ is Lipschitz}\}.$$

We define

$$\mathfrak{K}_2 := \{v : -M \leq \mathcal{D}_t^\alpha v \leq f \text{ on } [0, 2] \text{ and } u_k \Rightarrow v \text{ with } u_k \leq v \text{ and } u_k \in \mathfrak{K}_2\}.$$

We note that any $v \in \mathfrak{K}_2$ is continuous since it is the uniform limit on a compact set of Lipschitz continuous functions.

We now show that \mathfrak{K}_1 is nonempty, and hence \mathfrak{K}_2 is also nonempty. From the theory of ordinary differential equations [9], we solve

$$\partial_t^\alpha g(t) = h(t)/\lambda \leq f(t)/\lambda,$$

with h smooth and strictly decreasing. Then g is strictly decreasing, and so

$$\mathcal{D}_t^\alpha g \leq M_\alpha^+ g(t) = \lambda \partial_t^\alpha g(t) = h(t) \leq f(t).$$

Thus g is a smooth subsolution to (5.2) and g can be approximated from below by itself. We choose $-M < \inf h/\lambda$. Thus \mathfrak{K}_2 is nonempty.

We now assign a partial ordering to \mathfrak{K}_2 with the natural assignment that $v_1 \leq v_2$ if $v_1 \leq v_2$ everywhere on $(-\infty, 0]$. From the comparison principle we have that u as in (5.2) is an upper bound for \mathfrak{K}_2 .

We will now show that $u \in \mathfrak{K}_2$. By Zorn's Lemma there exists a maximal element w . We will show that $w \equiv u$. If $\mathcal{D}_t^\alpha w \equiv \mathcal{D}_t^\alpha u$, then by comparison and uniqueness, $w \equiv u$. Suppose by way of contradiction that w is not identically u . Then w is not a supersolution of (5.2), and so there exists a $t_0 \in [0, 2]$ and a Lipschitz function ψ with $\psi(t_0) = w(t_0)$ and $\psi \leq w$ on $[t_0 - \delta, t]$ for some $\delta > 0$ such that $\mathcal{D}_t^\alpha \phi(t_0) < f(t_0)$ where

$$\phi(t) := \begin{cases} \psi(t) & \text{if } t \in (t_0 - \delta], \\ w(t) & \text{if } t < t_0 - \delta. \end{cases}$$

Then from Proposition 2.3 we can evaluate $\mathcal{D}_t^\alpha w(t_0)$ classically and $\mathcal{D}_t^\alpha w(t_0) < f(t_0)$. Because $w \in \mathfrak{K}_2$ there exists Lipschitz subsolutions $w_k \rightarrow w$ uniformly from below. Then there exists k large enough so that

$$\phi_k(t) := \begin{cases} \psi(t) & \text{if } t \in (t_0 - \delta], \\ w_k(t) & \text{if } t < t_0 - \delta. \end{cases}$$

satisfies $M_\alpha^+ \phi_k < f$ on $[t_0 - \delta_1, t_0]$ for some $0 < \delta_1 < \delta$. We now consider two different situations. If $w_k(t_0) = w(t_0)$, then for k large enough, $\mathcal{D}_t^\alpha w_k(t_0) < f(t_0)$ because $\mathcal{D}_t^\alpha w(t_0) < f(t_0)$ classically. Since w_k is Lipschitz, it follows from Lemma 5.1 that $\mathcal{D}_t^\alpha w_k < f$ in $[t_0 - \delta_2, t_0 + \delta_2]$. Then since $\mathcal{D}_t^\alpha(w_k + \eta_\epsilon) = \mathcal{D}_t^\alpha w_k + \mathcal{D}_t^\alpha \eta_\epsilon$, then $w_k + \eta_\epsilon$ is a Lipschitz subsolution for ϵ small enough, and $w_k(t_0) + \eta_\epsilon(t_0) > w(t_0)$. Now the $\max\{w_k + \eta_\epsilon, w\} \in \mathfrak{K}_2$, and this contradicts the maximality of w .

If in the second situation, $w_k(t_0) < w(t_0)$, then we extend ϕ_k to the right of t_0 by $\phi_k(t_0) = -M_1(t_0 - t) + w(t_0)$. From Lemma 5.1, for M_1 large enough, $\mathcal{D}_t^\alpha \phi_k \leq f$

for all $t \in [-2, 0]$, and $\mathcal{D}_t^\alpha \phi_k < f$ in $[t_0 - \delta_3, t_0 + \delta_3]$. We let $\tilde{\phi}_k := \max\{\phi_k, w_k\}$ and note that $\tilde{\phi}_k \in \mathfrak{K}_1$ and Lipschitz, and $\mathcal{D}_t^\alpha \phi_k < f$ in $[t_0 - \delta_4, t_0 + \delta_4]$ for some $\delta_4 > 0$ with $\tilde{\phi}_k(t_0) = w(t_0)$. Then as before we may take $\tilde{\phi}_k + \eta_\epsilon$ with ϵ small enough and obtain a contradiction. \square

In order to prove Hölder continuity of solutions to (1.12) we will follow the method presented in [15]. One of the main ingredients is to solve an ordinary differential equation in time. We begin with the following.

Lemma 5.3. *Let C_1 be a fixed constant. Let $g(t)$ be a continuous function on $[-2, 0]$. There exists a continuous viscosity solution $m(t)$ in $[-2, 0]$ to*

$$\mathcal{D}_t^\alpha m(t) = C_1 m(t) + g(t),$$

with $m(t) = 0$ for $t \leq -2$.

Proof. Let $\beta_1 < \beta$ for β as in Theorem 1.1. From Theorem 3.5, for $v \in C^{\beta_1}([0, 2])$ there exists a solution $h(t)$ to

$$\mathcal{D}_t^\alpha h(t) = C_1 v(t) + g(t),$$

with $h(t) = 0$ for $t \leq -2$. From Theorem 1.1 we have that

$$\|h\|_{C^{0,\beta}} \leq C \|C_1 v + g\|_{L^\infty}.$$

Since $C^{0,\beta_1}([-2, 0])$ is compactly contained in $C^{0,\beta}([-2, 0])$ we have a compact mapping $M : v \rightarrow h$ from $C^{0,\beta_1}([-2, 0])$ into itself. From Corollary 11.2 in [10] it follows that there is a fixed point $m(t)$ which is a viscosity solution from Lemma 2.4. \square

As in [15] we will utilize an ordinary differential equation to capture information backwards in time. We consider the fractional ordinary differential equation

$$\begin{cases} m(-2) = 0 & \text{for } t \leq -2 \\ \mathcal{D}_t^\alpha m(t) = c_0 |\{x \in B_1 : u(x, t) < 0\}| - C_1 m(t) & \text{for } -2 < t \leq 0. \end{cases} \quad (5.3)$$

We would like to use m as a test function for a viscosity solution. However, since the right hand side is not continuous, we cannot apply Lemma 5.3 to obtain the existence of m . Furthermore, the solution m may not be Lipschitz and therefore not a valid test function. To overcome these two issues we obtain a Lipschitz subsolution to (6.3). We consider $|\{x \in B_1 : u(x, t) < 0\}|$ rather than $|\{x \in B_1 : u(x, t) \leq 0\}|$ because we may easily approximate the former from below by smooth functions. We accomplish this by considering $g(x, t, \epsilon) := \min\{\epsilon^{-1} \max\{0, -u\}, 1\}$. We then let

$$G(t, \epsilon) = \int_{B_1} g(x, t, \epsilon) \, dx.$$

Now $G(t, \epsilon)$ is continuous in t , and $0 \leq G(t, \epsilon) \leq |\{x \in B_1 : u(x, t) < 0\}|$, and $G(t, \epsilon) \rightarrow |\{x \in B_1 : u(x, t) < 0\}|$ as $\epsilon \rightarrow 0$. Since $G(t, \epsilon)$ is continuous, from Lemma 5.3, we may solve

$$\begin{cases} m(-2) = 0 & \text{for } t \leq -2 \\ \mathcal{D}_t^\alpha m(t) = c_0 G(t, \epsilon) - C_1 m(t) & \text{for } -2 < t \leq 0. \end{cases}$$

Lemma 5.4. *Let $\alpha_0 \leq \alpha < 1$. Assume that the kernel for \mathcal{D}_t^α satisfies (1.8). Let m be a solution to $\mathcal{D}_t^\alpha m = c_0 f(t) - C_1 m$ with $m(t) = 0$ for $t \leq -2$, $0 \leq f \leq |B_1|$, and*

$$\int_{-2}^{-1} f(t) dt \geq \mu. \quad (5.4)$$

Then there exists two moduli of continuity $\omega_1(c_0\mu), \omega_2(C_1^{-1})$ with ω_i increasing and $\omega_i(s) > 0$ for $s > 0$ and depending only on $\lambda, \Lambda, |B_1|, \alpha_0$ so that

$$m(t) \geq \omega_1(c_0\mu)\omega_2(C_1^{-1}) \quad \text{for } -1 \leq t \leq 0. \quad (5.5)$$

Proof. From Lemma 4.12 any such solution m is Hölder continuous. We claim that $m > 0$ for $t \in [-1, 0]$. Suppose by way of contradiction that there exists $t_0 \in [-1, 0]$ such that $m(t_0) \leq 0$. Let m achieve its minimum at t_1 . Then m is touched from below at t_1 by the constant function $m(t_1)$, and so by Propositions 2.2 and 2.3 we may evaluate $\mathcal{D}_t^\alpha m$ at t_1 and

$$\int_{-\infty}^{t_1} [m(t_1) - m(s)] \mathcal{K}(t, s) ds \geq f(t_1) \geq 0.$$

But since $m(t_1) \leq 0$ and m achieves a minimum at t_1 we have

$$0 \geq \int_{-\infty}^{t_1} [m(t_1) - m(s)] \mathcal{K}(t, s) ds.$$

Then $m(t) = 0$ for $t \leq t_1$. But then since $f(t) = 0$ for $t \leq t_1$. But this contradicts the assumption (5.4). Then $m(t) > 0$ for $-1 \leq t \leq 0$.

Let us fix $c_0, C_1, \mu > 0$. Suppose by way of contradiction that there exists m_k, α_k, f_k all satisfying the assumptions of the Lemma, but

$$\inf_{[-1, 0]} m_k \rightarrow 0$$

as $k \rightarrow \infty$. From Lemma 4.12, we have $m_k \rightarrow m_0$ in $C^{0, \beta_1}([-2, 0])$ for any $\beta_1 < \beta$. Then there exists $t_1 \in [-1, 0]$ such that $m_0(t_1) = 0$.

We now claim that $m_0(t_0) > 0$ for some $t_0 \in [-2, -1]$. Suppose by way of contradiction that $m_0 \equiv 0$ in $[-2, -1]$. From Corollary 3.7, we have that

$$\begin{aligned} C\Lambda \sup_{[-2, -1]} m_k &\geq \frac{\Lambda}{\Gamma(1 - \alpha_k)} \int_{-2}^{-1} \frac{m_k(t)}{(-1 - t)^\alpha} dt \\ &\geq \int_{-2}^{-1} c_0 f_k(t) - C_1 m_k dt \\ &\geq c_0 \mu - \int_{-2}^{-1} C_1 m_k dt. \end{aligned}$$

Letting $k \rightarrow \infty$ we obtain that $0 \geq \mu$. Which is a contradiction, and therefore the claim that m_0 is not identically zero is true.

We now consider two different cases. First assume that for a subsequence $\alpha_k \rightarrow \alpha_1 < 1$. Let t_2 be the first point after t_0 such that $m_0(t_2) = 0$. Because $m_k \rightarrow m_0$ uniformly, and since $m_0(t_0) > 0$, we may choose $\psi \geq 0$ smooth with $\psi(t) = 0$ in a neighborhood of t_2 and $\psi(t_0) > 0$ and also satisfying $m_k \geq \psi$ for all k . Now $M_\alpha^+ \psi(t) \leq -\delta_1$ for $t \in [t_2 - \delta_2]$ and $\alpha \in (\alpha_1 - \delta_3, \alpha + \delta_3)$. We let ϵ_k be such that

$\psi + \epsilon_k \leq m_k$ on $[t_2 - \delta_2, t_2]$ and $\psi(t_k) + \epsilon_k = m_k(t_k)$ for some $t_k \in [t_2 - \delta_2, t_2]$. We define

$$\phi_k := \begin{cases} \psi + \epsilon_k & \text{if } t \geq t_2 - \delta_2 \\ m_k & \text{if } t < t_2 - \delta_2. \end{cases}$$

Then

$$M_{\alpha}^+ \phi_k(t_k) \geq \mathcal{D}_t^{\alpha_k} \phi_k(t_k) \geq -C_1 m_k(t_k).$$

Since $m_0(t) > 0$ for $t \in (t_2 - \delta_2, t_2)$, as $k \rightarrow \infty$, we have $t_k \rightarrow t_2$ and $\epsilon \rightarrow 0$. Then

$$\lim_{k \rightarrow \infty} M_{\alpha_k}^+ \phi_k(t_k) \geq \lim_{k \rightarrow \infty} -C_1 m_k(t_k) = 0.$$

But we also have

$$\lim_{k \rightarrow \infty} M_{\alpha_k}^+ \phi_k(t_k) \leq M_{\alpha_1}^+ \psi(t_2) \leq -\delta_1 < 0.$$

This is a contradiction to the first case.

We now consider the case in which $\alpha_k \rightarrow 1$. For a further subsequence there exists f_0 such that f_k converges to f_0 in weak star L^∞ , so that

$$\int_{-2}^{-1} f_0(t) \geq \mu.$$

Then from Corollary 3.7 we have that

$$\frac{\lambda \alpha_k}{\Gamma(1 - \alpha_k)} \int_{-2}^t m_k(s) \, ds \leq \int_{-2}^t f_k(s) - C_1 m_k(s) \, ds \leq \frac{\Lambda \alpha_k}{\Gamma(1 - \alpha_k)} \int_{-2}^t m_k(s) \, ds.$$

Since $m_k \rightarrow m_0$ uniformly and since [9] for any continuous function h we have

$$\frac{\alpha_k}{\Gamma(1 - \alpha_k)} \int_{-2}^t \frac{h(s)}{(t - s)^{\alpha_k}} \, ds \rightarrow h(t),$$

as $\alpha_k \rightarrow 1$, then we obtain as $k \rightarrow \infty$ the inequality

$$\lambda m_0(t) \leq \int_{-2}^t f_0(s) - C_1 m_0(s) \, ds \leq \Lambda m_0(t).$$

Then there exists $\lambda \leq g(t) \leq \Lambda$ such that

$$g(t) m_0(t) = \int_{-2}^t f_0(s) - C_1 m_0(s) \, ds.$$

Then gm_0 is Lipschitz continuous since f_0 is bounded. Furthermore, we have that

$$g(t) m_0(t) \geq \int_{-2}^t f_0(s) - C_1 \lambda^{-1} g(s) m_0(s) \, ds.$$

Then gm_0 is a supersolution and from the theory of ordinary differential equations, $gm_0 \geq \tilde{g}$ with \tilde{g} solving

$$\tilde{g}(t) = \int_{-2}^t f_0(s) - C_1 \lambda^{-1} \tilde{g}(s) \, ds.$$

Since f_0 is not identically zero, then $\tilde{g} > 0$ on $[-1, 0]$. It follows that $gm_0 > 0$ and hence $m_0 > 0$ on $[-1, 0]$ as well. This is a contradiction to the second case. Then for fixed c_0, C_1, μ , there exists $\delta_4 > 0$ depending on $c_0, C_1, \mu, \lambda, \Lambda, \alpha_0$ such that for any solution satisfying the assumptions in the statement of the lemma, we have that $u \geq \delta_4$ in $[-1, 0]$. We then obtain a modulus of continuity as stated in the Lemma. \square

6. HÖLDER CONTINUITY

In this section we follow the method used in [15] to prove our main result. We will need the following Proposition to account for the growth in the tails.

Lemma 6.1. *Let u be a continuous function, $u \leq 1$ in $(\mathbb{R}^n \times [-2, 0]) \cup (B_2 \times [-\infty, 0])$, which satisfies the inequality in the viscosity sense in $B_2 \times [-2, 0]$*

$$\mathcal{D}_t^\alpha u - M^+ u \leq \epsilon_0. \quad (6.1)$$

with $\alpha_0 \leq \alpha < 1$. Assume also that

$$|\{u \leq 0\} \cap (B_1 \times [-2, -1])| \geq \mu. \quad (6.2)$$

Then if ϵ_0 is small enough there exists $\theta > 0$ such that $u \leq 1 - \theta$ in $B_1 \times [-1, 0]$. The maximum value of ϵ_0 as well as θ depend only on $\alpha_0, \lambda, \Lambda, n$ and σ .

Proof. We first mention that it is sufficient to prove the Lemma under the assumption

$$|\{u < 0\} \cap (B_1 \times [-2, -1])| \geq \mu.$$

For if u satisfies (6.1) and (6.2), then $u - c$ for any positive constant c will satisfy (6.1) as well as the inequality above. Then $u - c \leq 1 - \theta$ in $B_1 \times [-1, 0]$ independent of c and so letting $c \rightarrow 0$ one obtains $u \leq 1 - \theta$.

We consider the fractional ordinary differential equation $m : (-\infty, 0] \rightarrow \mathbb{R}$

$$\begin{aligned} m(t) &= 0 & \text{for } t \leq -2 \\ \mathcal{D}_t^\alpha m(t) &= c_0 f(t) - C_1 m(t) & \text{for } t > -2, \end{aligned} \quad (6.3)$$

Where $f(t)$ is a smooth approximation from below of $|\{x \in B_1 : u(x, t) < 0\}|$. From the hypothesis and Lemma 5.4, we can choose an approximation f such that

$$m(t) \geq \omega_1(c_0 \mu) \omega_2(C_1^{-1})/2 > 0,$$

for $t \in [-1, 0]$. By Lemma 5.2 we can approximate m uniformly from below by a Lipschitz function g such that

$$\begin{aligned} g(-2) &= 0 \\ \mathcal{D}_t^\alpha g(t) &\leq c_0 f(t) - C_1 m(t), \end{aligned}$$

and

$$g(t) \geq \omega_1(c_0 \mu) \omega_2(C_1^{-1})/4 > 0,$$

for $t \in [-1, 0]$. We utilize the function $g(t)$ which is not just a viscosity solution but also a classical solution since it is Lipschitz. We can then calculate $\mathcal{D}_t^\alpha g$ everywhere classically. Furthermore, g is allowed as a test function for touching above or below for viscosity solutions.

We want to show that $u \leq 1 - g(t) + \epsilon_0 c_\alpha 2^\alpha$ if c_0 is small and C_1 is large. We can then set $\theta = \omega_1(c_0 \mu) \omega_2(C_1^{-1})/4$ for ϵ_0 small to obtain the result of the Lemma. We pick the constant c_α such that $\partial_t^\alpha c_\alpha (2+t)_+^\alpha = 1$ for $t > -2$, and note [9] that c_α is uniform as $\alpha \rightarrow 1$. Let $\beta : \mathbb{R} \rightarrow \mathbb{R}$ be a fixed smooth nonincreasing function such that $\beta(x) = 1$ if $x \leq 1$ and $\beta(x) = 0$ if $x \geq 2$. Let $b(x) = \beta(|x|)$. Where $b = 0$ we have $M^- b > 0$. Since b is smooth $M^- b$ is continuous and it remains positive for b small enough ([15]). Thus there exists β_1 such that $M^- b \geq 0$ if $b \leq \beta_1$.

Assume that there exists some point $(x, t) \in B_1 \times [-1, 0]$ such that

$$u(x, t) > 1 - g(t) + \epsilon_0 \lambda^{-1} c_\alpha (2+t)_+^\alpha.$$

We will arrive at a contradiction by looking at the maximum of the function

$$w(x, t) = u(x, t) + g(t)b(x) - \epsilon_0 \lambda^{-1} c_\alpha (2+t)_+^\alpha$$

We assume there exists a point in $B_1 \times [-1, 0]$ where $w(x, t) > 1$. Let (x_0, t_0) be the point that realizes the maximum of w :

$$w(x_0, t_0) = \max_{\mathbb{R}^n \times (-\infty, 0]} w(x, t).$$

This maximum is larger than 1, and so it must be achieved when $t > -2$ and $|x| < 2$.

Let $\phi(x, t) := w(x_0, t_0) - g(t)b(x) + \epsilon_0 \lambda^{-1} c_\alpha (2+t)_+^\alpha$. We remark that $\phi(x, t) = \phi(x, -2)$ for $t \leq -2$, and ϕ touches u from above at the point (x_0, t_0) . We define

$$v(x, t) := \begin{cases} \phi(x, t) & \text{if } x \in B_r \\ u(x, t) & \text{if } x \notin B_r. \end{cases}$$

Then from the definition of viscosity solution we have

$$\partial_t^\alpha v - M^+ v \leq \epsilon_0 \quad \text{at } (x_0, t_0). \quad (6.4)$$

We have that

$$\begin{aligned} \mathcal{D}_t^\alpha v(x_0, t_0) &= \mathcal{D}_t^\alpha (-g(t_0)b(x_0) + \epsilon_0 \lambda^{-1} c_\alpha (2+t)_+^\alpha) \\ &\geq (C_1 m(t) - c_0 f(t))b(x_0) + \epsilon_0 \lambda^{-1} M_\alpha^- (2+t)_+^\alpha \\ &= (C_1 m(t) - c_0 f(t))b(x_0) + \epsilon_0 \lambda^{-1} \lambda \partial_t^\alpha (2+t)_+^\alpha \\ &= (C_1 m(t) - c_0 f(t))b(x_0) + \epsilon_0 \\ &\geq (C_1 m(t) - c_0 |\{x \in B_1 : u(x, t_0) < 0\}|)b(x_0) + \epsilon_0 \\ &\geq (C_1 m(t) - c_0 |\{x \in B_1 : u(x, t_0) \leq 0\}|)b(x_0) + \epsilon_0. \end{aligned}$$

Then

$$\begin{aligned} \epsilon_0 &\geq \mathcal{D}_t^\alpha v(x_0, t_0) - M^+ v(x_0, t_0) \\ &\geq (C_1 m(t) - c_0 |\{x \in B_1 : u(x, t_0) \leq 0\}|)b(x_0) + \epsilon_0 - M^+ v(x_0, t_0), \end{aligned}$$

or

$$0 \geq (C_1 m(t) - c_0 |\{x \in B_1 : u(x, t_0) \leq 0\}|)b(x_0) - M^+ v(x_0, t_0). \quad (6.5)$$

Now exactly as in [15] we obtain the following bound for $G := \{x \in B_1 \mid u(x, t_0) \leq 0\}$,

$$M^+ v(x_0, t_0) \leq -m(t_0)M^- b(x_0, t_0) - c_0 |G \setminus B_r| \quad (6.6)$$

for some universal constant c_0 . This is how we choose c_0 in the fractional ordinary differential equation. We now look at two different cases and obtain a contradiction in both. Suppose $b(x_0) \leq \beta_1$. Then $M^- b(x_0) \geq 0$, and so from (6.6)

$$M^+ v(x_0, t_0) \leq -c_0 |G \setminus B_r|.$$

Combining the above inequality with (6.5), we obtain

$$0 \geq (-c_0 |\{x \in B_1 : u(x, t) \leq 0\}| + C_1 m(t))b(x_0) + c_0 |G \setminus B_r|.$$

For any $C_1 > 0$ this will be a contradiction by taking r small enough.

Now suppose $b(x_0) > \beta_1$. Since b is a smooth compactly supported function, there exists C such that $|M^- b| \leq C$. We then have from (6.6) the bound

$$M^+ v(x_0, t_0) \leq C m(t_0) - c_0 |G \setminus B_r|$$

and inserting this in (6.5) we have

$$0 \geq (-c_0|\{x \in B_1 : u(x, t) \leq 0\}| + C_1 m(t)) b(x_0) - C m(t_0) + c_0 |G \setminus B_r|$$

Letting $r \rightarrow 0$ we obtain

$$\begin{aligned} 0 &\geq c_0(1 - b(x_0))|G| + (C_1 b(x_0) - C)m(t_0) \\ &\geq c_0(1 - b(x_0))|G| + (C_1 \beta_1 - C)m(t_0). \end{aligned}$$

Choosing C_1 large enough we obtain a contradiction. \square

We now define

$$Q_r := B_r \times [-r^{2\sigma/\alpha}, 0],$$

and note the rescaling property that if $v(x, t) = u(rx, r^{2\sigma/\alpha}t)$, then

$$\tilde{\mathcal{D}}_t^\alpha v(x, t) - M_\sigma^\pm v(x, t) = r^{2\sigma} \left(\mathcal{D}_t^\alpha u(rx, r^{2\sigma/\alpha}t) - M_\sigma^\pm u(rx, r^{2\sigma/\alpha}t) \right)$$

where if $\mathcal{K}(t, s)$ is the kernel for \mathcal{D}_t^α , then $\tilde{\mathcal{D}}_t^\alpha$ has kernel

$$\frac{\mathcal{K}(rt, rs)}{r^{1+\alpha}}$$

which will also satisfy (1.8) and (1.9). For the next three results we fix $r = \min\{4^{-1}, 4^{-\alpha/2\sigma}\}$. We will need the following Proposition to bound the tails.

Proposition 6.2. *Let $h(t) = \max\{2|rt|^\nu - 1, 0\}$ with $r = \min\{4^{-1}, 4^{-\alpha/2\sigma}\}$. If $t_1 \leq 0$ and $\nu < \alpha$ then*

$$0 \geq \mathcal{D}_t^\alpha h(t_1) \geq -\lambda c_{\alpha, \nu}$$

where $c_{\alpha, \nu}$ is a constant depending only on α and ν but for fixed ν remains uniform as $\alpha \rightarrow 1$.

Proof. Now

$$0 \geq \mathcal{D}_t^\alpha h(t_1) \geq M_\alpha^- h(t_1) = \lambda \partial_t^\alpha h(t_1).$$

From [1] we have

$$\partial_t^\alpha h(t_1) \geq c_{\alpha, \nu},$$

which for fixed ν remains uniform as $\alpha \rightarrow 1$. Combining the above two inequalities, the Proposition is proven. \square

Lemma 6.3. *Let u be a bounded continuous function which satisfies the following two inequalities in the viscosity sense in Q_1*

$$\begin{aligned} \mathcal{D}_t^\alpha u - M^+ u &\leq \epsilon_0/2, \\ \mathcal{D}_t^\alpha u - M^- u &\geq -\epsilon_0/2. \end{aligned} \tag{6.7}$$

Let the kernel $\mathcal{K}(t, s)$ of \mathcal{D}_t^α satisfy (1.8) and (1.9) with $0 < \alpha_0 \leq \alpha < 1$. Then there are universal constants $\theta > 0$ and $\nu > 0$ depending only on $n, \sigma, \Lambda, \lambda, \alpha_0$ such that if

$$\begin{aligned} |u| &\leq 1 \quad \text{in } B_1 \times [-1, 0] \\ |u(x, t)| &\leq 2|rx|^\nu - 1 \quad \text{in } (\mathbb{R}^n \setminus B_1) \times [-1, 0] \\ |u(x, t)| &\leq 2|rt|^\nu - 1 \quad \text{in } B_1 \times (-\infty, -1], \end{aligned}$$

with $r = \min\{4^{-1}, 4^{-\alpha/2\sigma}\}$, then

$$\text{osc}_{Q_r} u \leq 1 - \theta$$

Proof. We consider the rescaled version

$$\tilde{u}(x, t) := u(r^{-1}x, r^{-2\sigma/\alpha}t).$$

The function \tilde{u} will stay either positive or negative in half of the points in $B_1 \times [-2, -1]$. Let us assume that $\{\tilde{u} \leq 0\} \cap (B_1 \times [-2, -1]) \geq |B_1|/2$. Otherwise we can repeat the proof for $-\tilde{u}$. We would like to apply Lemma 6.1. To do so we would need $\tilde{u} \leq 1$. We consider $v := \min\{1, \tilde{u}\}$. Inside $Q_{r^{-1}}$ we have $v = \tilde{u}$. The error comes only from the tails in the computations. Exactly as in [15] we obtain for κ small enough

$$-M^+v \leq -M^+\tilde{u} + \epsilon_0/4.$$

From Proposition 6.2 we have for small enough κ that

$$\mathcal{D}_t^\alpha v \leq \mathcal{D}_t^\alpha \tilde{u} + \epsilon_0/4.$$

Thus

$$\mathcal{D}_t^\alpha v - M^+v \leq \epsilon_0.$$

We now apply Lemma 6.1 to v and rescale back to conclude the proof. \square

We are now able to give the proof of our main result.

Theorem 1.2. We first choose $\kappa < \nu$ for ν as in Lemma 6.3 Let $(x_0, t_0) \in Q_1$. We consider the rescaled function

$$v(x, t) = \frac{u(x_0 + x, t + t_0)}{\|u\|_{L^\infty} + \epsilon_0^{-1}\|f\|_{L^\infty}},$$

and note that $|v| \leq 1$ and v is a solution to

$$\begin{aligned} \mathcal{D}_t^\alpha v - M_\sigma^+ v &\leq \epsilon_0 \\ \mathcal{D}_t^\alpha v - M_\sigma^- v &\geq -\epsilon_0. \end{aligned}$$

in $B_2 \times [-1, 0]$. We let $r = \min\{4^{-1}, 4^{-\alpha/2\sigma}\}$ and the estimate will follow as soon as we show

$$\text{osc}_{Q_{r^k}} v \leq 2r^{\kappa k}. \quad (6.8)$$

Estimate (6.8) will be proven by constructing two sequences $a_k \leq v \leq b_k$ in Q_{r^k} , $b_k - a_k = 2r^{\kappa k}$ with a_k nondecreasing and b_k nonincreasing. The sequence is constructed inductively.

Since $|v| \leq 1$ everywhere, we can start by choosing some $a_0 \leq \inf v$ and $b_0 \geq \sup v$ so that $b_0 - a_0 = 2$. Assuming now that the sequences have been constructed up to the value k we scale

$$w(x, t) = (v(r^k x, r^{2k\sigma/\alpha} t) - (a_k + b_k)/2)r^{-\kappa k}.$$

We then have

$$\begin{aligned} |w| &\leq 1 \quad \text{in } Q_1 \\ |w| &\leq 2r^{-\kappa k} - 1 \quad \text{in } Q_{r^{-k}}. \end{aligned}$$

and so

$$\begin{aligned} |w(x, t)| &\leq 2|x|^\nu - 1 \quad \text{for } (x, t) \in B_1^c \times [-1, 0] \\ |w(x, t)| &\leq 2|t|^\nu - 1 \quad \text{for } (x, t) \in B_1 \times (-\infty, -1). \end{aligned}$$

Notice also that w has new right hand side bounded by

$$\epsilon_0 r^{k(\kappa - 2\sigma)}$$

which is strictly smaller than ϵ_0 for $\kappa < 2\sigma$. For κ small enough we can apply Lemma 6.3 to obtain

$$\operatorname{osc}_{Q_r} w \leq 1 - \theta.$$

Then if κ is chosen smaller than the κ in Lemma 6.3 and also so that $1 - \theta \leq r^\kappa$, then this implies

$$\operatorname{osc}_{Q_{r^{k+1}}} w \leq r^{\kappa(k+1)}$$

so we can find a_{k+1} and b_{k+1} and this finishes the proof. \square

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